Department of Basic Sciences

MATRICES

ENGINEERING MATHEMATICS

Subject code: BT-102

Subject Name: (MATHEMATICS-I)

Semester: I

Faculty Name : Akhilesh Jain

Module 5: Matrices: Rank of a Matrix, Solution of Simultaneous Linear Equations by Elementary Transformation, Consistency of Equation, Eigen Values and Eigen Vectors, Diagonalization of Matrices, Cayley-Hamilton theorem and its applications to find inverse

1.1 Introduction

Matrices provide a very powerful tool for dealing with linear models. In this lesson we give a method of elementary row transformations and evaluation of inverse of matrices are discussed. Some examples are also provided.

1.2 Matrix: The arrangement of set of elements in the form of rows and columns is called as Matrix. The elements of the matrix being Real (or) Complex Numbers.

1.3 Order of the Matrix The number of rows and columns represents the order of the matrix. It is denoted by m x n , where m is number of rows and n is number of columns.

Note: Matrix is a system of representation and it does not have any Numerical value.

1.4 Types of Matrices :

Rectangular Matrix: A matrix is said to be rectangular, if the number of rows and number of columns are not equal.

Example: A = $\begin{bmatrix} 3 & 4 & 1 \\ -6 & 2 & 9 \end{bmatrix}$ is a rectangular matrix

1.5 Square Matrix : A matrix is said to be square, if the number of rows and number of columns are equal.

Example: A = $\begin{bmatrix} 1 & -2 \\ 6 & 8 \end{bmatrix}$ is a 2 x 2 Square matrix

1.6 Row Matrix : A matrix is said to be row matrix, if it contains only one row.

Example: A = $\begin{bmatrix} 1 & 5 & 4 & 5 \end{bmatrix}$ is a row matrix

1.7 Column Matrix : A matrix is said to be column m atrix, if it contains only one column

Example: A = $\begin{bmatrix} -4 \\ 5 \\ 2 \end{bmatrix}$ is a column matrix

1.8 Diagonal Matrix : A Square matrix is said to be diagonal matrix, if all the elements except principle diagonal elements are zeros.

- ❖ The elements on the diagonal are known as principle diagonal elements.
  - ❖ The diagonal matrix is represented by D = dia $\{a_{11}, a_{22}, a_{33}, \dots a_{nn}\}$ its diagonal elements.

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1.9 Rank of a matrix: The order of non-vanishing minor is called the rank of a matrix

or

A Positive number ρ is said to be the rank of a matrix *A* if

(i) There exists at least one r^{th} order -rowed minor, whose value is not equals to zero

(ii) Every $(r+1)^{th}$ order –rowed minor of A is zero.

Rank of matrix is denoted by ρ(A)

Note :

1. The rank of a matrix is always unique.
2. The rank of a zero matrix is always zero.
3. The rank of a non –singular matrix of order "*n*" is equals to "*n*".
4. The rank of a singular matrix of order "*n*" is less than "*n*".
5. The rank of a unit matrix of order "*n*" is equals to "*n*".
6. If *A* is a matrix of order $m \times n$, then the rank of $A = \rho(A) \le \{m,n\}$.
7. Rank of a matrix = rank of the transpose matrix *i.e.* $\rho(A) = \rho(A')$

Example: Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$ Since $|A|_{3x3} = 0$

But $I_r = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_r & 0 \end{bmatrix} = \begin{bmatrix} I_r \\ 0 \end{bmatrix}$, hence It is of rank ρ = 2 .

1.10 The elementary operations for matrices. The following operations, performed on a matrix, do not change either its order or its rank.

1. Interchanging any two rows or any two columns.
2. Multiplying any row or column by a non-zero constant.
3. Adding to any row a constant times another row or adding to any column a constant times another column.

We denote the different operations as follows:

1. R_{ij} (Or $R_i \leftrightarrow R_j$) interchange of the i^{th} and j^{th} rows
2. C_{ij} (Or $C_i \leftrightarrow C_j$) interchange of the i^{th} and j^{th} columns
3. $R_i(k)$ multiplication of the i^{th} row by the non-zero constant *k* .
4. $C_i(k)$ multiplication of the i^{th} column by the non-zero constant *k*
5. $R_{ij}(k)$ (Or $R_i + k\,R_j$) addition to the i^{th} row the product of *k* times the j^{th} row

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6. $C_{ij}(k)$ (Or $C_i + k\, C_j$) addition to the i^{th} column the product of k times the j^{th} column

These operations are termed the **elementary operations** or **elementary transformations**.

1.11 Equivalent matrices: Two $m \times n$ matrices are called equivalent if one can be obtained from the other by a sequence of elementary operations.

Equivalent matrices have the same order and the same rank.

1.12 Normal form (or) canonical form:

Any matrix of rank r > 0 can be reduced by elementary row and column operations to a canonical form, referred to as its normal form, of one of the following four types: $I_r = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_r & 0 \end{bmatrix} = \begin{bmatrix} I_r \\ 0 \end{bmatrix}$

where I_r is the identity matrix of order **r .**

All matrices that reduce to the same normal form through elementary row and column transformations are equivalent. Two m x n matrices A and B of the same rank will reduce to the same normal form.

1.13 Echelon Form:

A matrix is an ***echelon matrix*** (or is in echelon form) if:

1. Any zero rows are at the bottom;
2. The number of zeros before the non zero element of a row is less than such zeros in the next row.
3. An upper-triangular matrix is a special case of an echelon matrix, and an echelon matrix is necessarily upper triangular.

Example: **A =** $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 7 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

1.14 Reduced row echelon form and elementary row operations:

In above example, the key to solve a system of linear equations is to transform the original augmented matrix to some matrix with some properties via a few elementary row operations. As a matter of fact, we can solve any system of linear equations by transforming the associate augmented matrix to a matrix in some form. The form is referred to as the reduced row echelon form.

Definition: A matrix in reduced row echelon form has the following properties:

1. All rows consisting entirely of 0 are at the bottom of the matrix.
2. For each nonzero row, the first entry is 1. The first entry is called a leading 1.
3. For two successive nonzero rows, the leading 1 in the higher row appears farther to the left than

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the leading 1 in the lower row.

4. If a column contains a leading 1, then all other entries in that column are 0.

Note: a **matrix is in row echelon form as the matrix has the first 3 properties.**

Example: $\begin{bmatrix} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

are the matrices in reduced row echelon form.

The matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is **not** in reduced row echelon form but **in row echelon form ,** since the matrix has the first 3 properties and all the other entries above the leading 1 in the third column are not 0.

The matrix $\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ are **not** in row echelon form (also **not** in reduced row echelon form) since the leading 1 in the second row is not in the left of the leading 1 in the third row and all the other entries above the leading 1 in the third column are not 0.

1.15 Definition of elementary row operation: There are 3 elementary row operations:

1. Interchange two rows
2. Multiply a row by some nonzero constant
3. Add a multiple of a row to another row.

1.16 Reduced row echelon form and elementary row operations:

In above motivating example, the key to solve a system of linear equations is to transform the original augmented matrix to some matrix with some properties via a few elementary row operations. As a matter of fact, we can solve **any** system of linear equations by **transforming** the associate augmented matrix to a matrix in some form. The form is referred to as the reduced row echelon form.

Definition: A matrix in reduced row echelon form has the following properties:

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1. All rows consisting entirely of 0 are at the bottom of the matrix.
2. For each nonzero row, the first entry is 1. The first entry is called a leading 1.
3. For two successive nonzero rows, the leading 1 in the higher row appears farther to the left than the leading 1 in the lower row.
4. If a column contains a leading 1, then all other entries in that column are 0.

Note: a matrix is in row echelon form as the matrix has the first 3 properties.

Example : The matrix $\begin{bmatrix} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

are the matrices in reduced row echelon form.

The matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is **not** in reduced row echelon form but **in row echelon form** , since the matrix has the first 3 properties and all the other entries above the leading 1 in the third column are not 0.

The matrix $\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ are **not** in row echelon form (also **not** in reduced row echelon form) since the leading 1 in the second row is not in the left of the leading 1 in the third row and all the other entries above the leading 1 in the third column are not 0.

1.17 Important results:

➢ *Every nonzero $m \times n$ matrix can be transformed to a unique matrix in reduced row echelon form via elementary row operations.*

➢ *If the augmented matrix $[A \vdots b]$ can be transformed to the matrix in reduced row echelon form $[C \vdots d]$ via elementary row operations, then the solutions for the linear system corresponding to $[C \vdots d]$ is exactly the same as the one corresponding to $[A \vdots b]$.*

1.18 Rank of the Matrix

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(i) Rank of a Matrix by Minor: Definition: A positive integer r is said to be the ***rank*** of a non-zero matrix A if it satisfies,

(i) the matrix A has at least one non-zero minor of order **r** and

(ii) all the minors of order greater than **r** are equal to zero.

The rank of the matrix A is denoted by ρ(A).

Alternatively, the rank of a matrix is the order of the largest non-zero minor of the matrix.

Example: What is the rank of $[A]=\begin{bmatrix}3 & 1 & 2\\ 2 & 0 & 5\\ 1 & 2 & 3\end{bmatrix}$?

Solution: The largest square sub matrix possible is of order 3 and that is $[A]$ itself. Since $\det(A)=-23\neq 0,$ the rank of $[A]=3.$

Example: What is the rank of $[A]=\begin{bmatrix}3 & 1 & 2\\ 2 & 0 & 5\\ 5 & 1 & 7\end{bmatrix}$?

Solution: The largest square sub matrix of $[A]$ is of order 3 and that is $[A]$ itself. Since $\det(A)=0$, the rank of $[A]$ is less than 3. The next largest square sub matrix would be a 2×2 matrix. One of the square sub matrices of $[A]$ is $[B]=\begin{bmatrix}3 & 1\\ 2 & 0\end{bmatrix}$

and $\det(B)=-2\neq 0$. Hence the rank of $[A]$ is 2. There is no need to look at other 2×2 sub matrices to establish that the rank of $[A]$ is 2.

Q1. Find rank of matrix $A=\begin{bmatrix}1 & 2 & 3\\ 1 & 4 & 2\\ 2 & 6 & 5\end{bmatrix}$ **Ans**. ρ(A)=2 **[Jan.03,Jan. 06, Dec. 07]**

Q2. Find the rank & nullity of the matrix $A=\begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 4\\ 3 & 4 & 5\end{bmatrix}$ **Ans**. ρ(A)=2 ,N(A)=1 **[Jan. 2007]**

Q3. Find the rank & nullity of the matrix $A=\begin{bmatrix}1 & 1 & 1\\ b+c & c+a & a+b\\ bc & ca & ab\end{bmatrix}$

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Ans.(i) If a=b=c, ρ(A)=1(ii) a=b≠c, ρ(A)=2(iii) a≠b≠c, ρ(A)=3

(ii) Rank of a Matrix by reducing a matrix in Echelon form):

The ***rank*** of matrix is equal to the number of non-zero rows of an echelon matrix**.**

(iii) Rank of a Matrix by reducing a matrix in Normal form:

Rank of the matrix is equal to order of the ***identity*** matrix in the normal form.

Remarks :

1. If ρ(A) = n, then A is non-singular.
2. If A is a square matrix of order n, then ρ(A) ≤ n.
3. If A is an m × n matrix then ρ(A) ≤ min. {m, n}.
4. If I_n is the unit matrix of order n, then $\rho(I_n) = n$.
5. A matrix and its transpose have the same rank.
6. If ρ(A) = 0 then A is a Null matrix.
7. If all minors of order r are equal to zero, then ρ(A) < r.
8. If A is not a null matrix, then ρ(A) ≥ 1.

Nullity of a square matrix = Rank of the matrix – Order of the Matrix

9. Find the rank & nullity of the matrix $A=\begin{bmatrix}1&3&4&3\\3&9&12&9\\1&3&4&1\end{bmatrix}$ **Ans**. ρ(A)=3 & nullity=1

[June 2004, June 2014, Nov. 2018]

10. Find the rank & nullity of the matrix $A=\begin{bmatrix}1&2&3&2\\2&3&5&1\\1&3&4&5\end{bmatrix}$ **Ans**. ρ(A)=2 & nullity is not defined.

[June 2006 , May 18 may19]

11. Find the rank & nullity of $A=\begin{bmatrix}2&1&-3&-6\\3&-3&1&2\\1&1&1&2\end{bmatrix}$ **Ans**. ρ(A)=3 & nullity is not defined.

[June 2009, March 2010]

12. Find the rank of the matrix $A=\begin{bmatrix}1&a&b&0\\0&c&d&1\\1&a&b&0\\0&c&d&1\end{bmatrix}$ **Ans.** ρ(A)=2 **[Jan. 2007,June 2015]**

13. Reduce matrix A to its **normal** form and then find its rank $A=\begin{bmatrix}-2&-1&-3&-1\\1&2&3&-1\\1&0&1&1\\0&1&1&-1\end{bmatrix}$ **Ans.** ρ(A)=2

[Dec .04]

14. Reduce matrix A to its **normal** form and then find its rank $A=\begin{bmatrix}2 & 3 & -1 & -1\\ 1 & -1 & -2 & -4\\ 3 & 1 & 3 & -2\\ 6 & 3 & 0 & -7\end{bmatrix}$ **Ans**. ρ(A)=3

[Dec. 03, June 05,June07,Dec 08, Dec.12, Dec.14]

15. Find the **normal** form and rank of the matrix $A=\begin{bmatrix}1 & 1 & 2\\ 1 & 2 & 3\\ 0 & -1 & -1\end{bmatrix}$ **[June 2017(CBCS)]**

16. Find the **normal** form and rank of the matrix $A=\begin{bmatrix}8 & 1 & 3 & 6\\ 0 & 3 & 2 & 2\\ -8 & -1 & -3 & 4\end{bmatrix}$ **Ans**. ρ(A)=3.

[June 2006, June 16(CBCS)]

17. Reduce matrix A in **normal** form and find its rank $A=\begin{bmatrix}2 & 3 & 4 & 5\\ 3 & 4 & 5 & 6\\ 4 & 5 & 6 & 7\\ 9 & 10 & 11 & 12\end{bmatrix}$ **Ans**. ρ(A)=2

[June 03, April 09,Dec.2011]

Find the rank of the matrix $A=\begin{bmatrix}1^2 & 2^2 & 3^2 & 4^2\\ 2^2 & 3^2 & 4^2 & 5^2\\ 3^2 & 4^2 & 5^2 & 6^2\\ 4^2 & 5^2 & 6^2 & 7^2\end{bmatrix}$ **Ans**. ρ(A)=3 **[Jan 08,June2011, June 2013]**

1.19 Row rank of a Matrix : The number of non-zero rows in the row reduced form of a matrix is called the row-rank of the matrix.

By the very definition, it is clear that row-equivalent matrices have the same row-rank. For a matrix A we write " row rank A " to denote the row-rank of A .

EXAMPLES

1. Determine the row-rank of the matrix $\begin{bmatrix}1 & 2 & 1\\ 2 & 3 & 1\\ 1 & 1 & 2\end{bmatrix}$

Solution: To determine the row-rank of A we proceed as follows.

$$\begin{bmatrix}1 & 2 & 1\\ 2 & 3 & 1\\ 1 & 1 & 2\end{bmatrix} \xrightarrow{R_{21}(-2),R_{31}(-1)} \begin{bmatrix}1 & 2 & 1\\ 0 & -1 & -1\\ 0 & -1 & 1\end{bmatrix}.$$

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$$\begin{bmatrix}1 & 2 & 1\\0 & -1 & -1\\0 & -1 & 1\end{bmatrix}\xrightarrow{R_2(-1),R_{32}(1)}\begin{bmatrix}1 & 2 & 1\\0 & 1 & 1\\0 & 0 & 2\end{bmatrix}.$$

$$\begin{bmatrix}1 & 2 & 1\\0 & 1 & 1\\0 & 0 & 2\end{bmatrix}\xrightarrow{R_3(1/2),R_{12}(-2)}\begin{bmatrix}1 & 0 & -1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}.$$

$$\begin{bmatrix}1 & 0 & -1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}\xrightarrow{R_{23}(-1),R_{13}(1)}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

The last matrix in Step 1d is the row reduced form of A which has 3 non-zero rows. Thus, row rank A = 3 .

2. Determine the row-rank of the matrix $\begin{bmatrix}1 & 2 & 1\\2 & 3 & 1\\1 & 1 & 0\end{bmatrix}$

Solution: Here we have

$$\begin{bmatrix}1 & 2 & 1\\2 & 3 & 1\\1 & 1 & 0\end{bmatrix}\xrightarrow{R_{21}(-2),R_{31}(-1)}\begin{bmatrix}1 & 2 & 1\\0 & -1 & -1\\0 & -1 & -1\end{bmatrix}.$$

$$\begin{bmatrix}1 & 2 & 1\\0 & -1 & -1\\0 & -1 & -1\end{bmatrix}\xrightarrow{R_2(-1),R_{32}(1)}\begin{bmatrix}1 & 2 & 1\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}.$$

Hence Row rank=2

1.19 Matrix algebra is used for solving systems of equations:

Matrix algebra is used to solve a system of simultaneous linear equations. In fact, for many mathematical procedures such as the solution to a set of nonlinear equations, interpolation, integration, and differential equations, the solutions reduce to a set of simultaneous linear equations. Let us illustrate with an example for interpolation.

A general set of m linear equations and n unknowns,

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$$a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = c_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = c_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots.$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = c_m$$

can be rewritten in the matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & . & . & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ c_m \end{bmatrix}$$

Denoting the matrices by $[A]$, $[X]$, and $[C]$, the system of equation is

$[A][X]=[C]$, where $[A]$ is called the coefficient matrix, $[C]$ is called the right hand side vector and $[X]$ is called the solution vector.

Sometimes $[A][X]=[C]$ systems of equations are written in the augmented form. That is

$$[A \vdots C] = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} & \vdots c_1 \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} & \vdots c_2 \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} & \vdots c_n \end{bmatrix}$$

A system of equations can be consistent or inconsistent. What does that mean?

A system of equations $[A][X]=[C]$ is consistent if there is a solution, and it is inconsistent if there is no solution. However, a consistent system of equations does not mean a unique solution, that is, a consistent system of equations may have a unique solution or infinite solutions (Figure 1).

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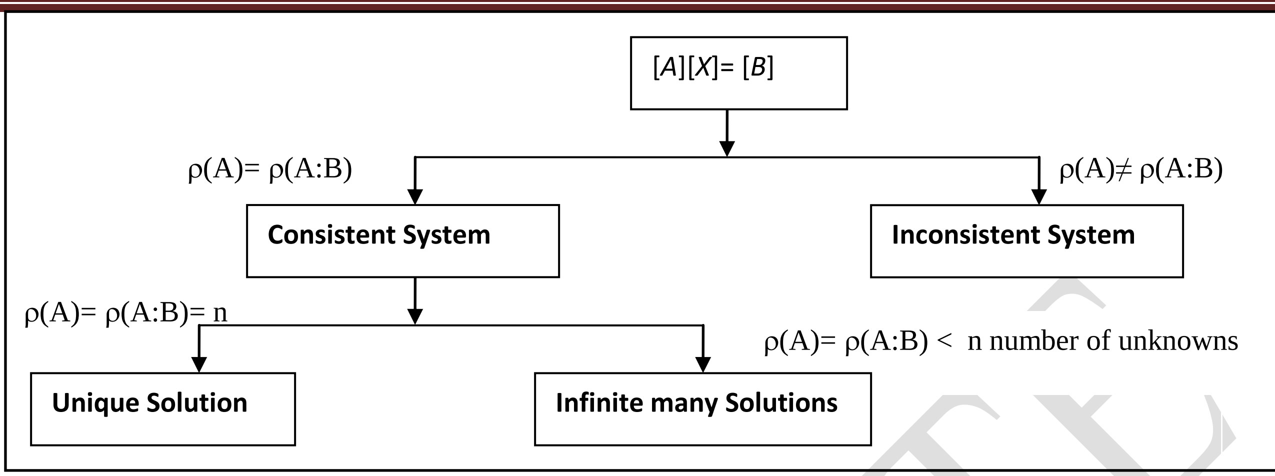

Figure. Consistent and inconsistent system of equations flow chart.

Example : Give examples of consistent and inconsistent system of equations**.**

Solution

a) The system of equations $\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$ is a consistent system of equations as it has a unique solution, that is, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

b) The system of equations $\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$ is also a consistent system of equations but it has infinite solutions as given as follows.

Expanding the above set of equations, $2x + 4y = 6, x + 2y = 3$

we can see that they are the same equation(Parallel Equations). Hence, any combination of (x, y) that satisfies $2x + 4y = 6$

is a solution. For example $(x, y) = (1,1)$ is a solution. Other solutions include $(x, y) = (0.5,1.25)$, $(x, y) = (0,\ 1.5)$, and so on.

c) The system of equations $\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$ is inconsistent as no solution exists.

Example: How do I now use the concept of rank to find if

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

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is a consistent or inconsistent system of equations?

Solution: The coefficient matrix is

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

and the right hand side vector is

$$[C] = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

The augmented matrix is

$$[B] = \left[\begin{array}{ccc:c} 25 & 5 & 1 & 106.8 \\ 64 & 8 & 1 & 177.2 \\ 144 & 12 & 1 & 279.2 \end{array}\right]$$

Since there are no square sub matrices of order 4 as $[B]$ is a 3×4 matrix, the rank of $[B]$ is at most 3. So let us look at the square sub matrices of $[B]$ of order 3; if any of these square sub matrices have determinant not equal to zero, then the rank is 3. For example, a sub matrix of the augmented matrix $[B]$ is

$$[D] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

has $\det(D) = -84 \neq 0$.

Hence the rank of the augmented matrix $[B]$ is 3. Since $[A] = [D]$, the rank of $[A]$ is 3. Since the rank of the augmented matrix $[B]$ equals the rank of the coefficient matrix $[A]$, the system of equations is consistent.

Example : Use the concept of rank of matrix to find if

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 89 & 13 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 284.0 \end{bmatrix}$$

is consistent or inconsistent?

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Solution : The coefficient matrix is given by $[A]=\begin{bmatrix}25 & 5 & 1\\ 64 & 8 & 1\\ 89 & 13 & 2\end{bmatrix}$

and the right hand side $[B]=\begin{bmatrix}106.8\\ 177.2\\ 284.0\end{bmatrix}$

The augmented matrix is $[A:B]=\begin{bmatrix}25 & 5 & 1 & :106.8\\ 64 & 8 & 1 & :177.2\\ 89 & 13 & 2 & :284.0\end{bmatrix}$

Since there are no square sub matrices of order 4 as [A:B] is a 4×3 matrix, the rank of the augmented [A:B] is at most 3. So let us look at square sub matrices of the augmented matrix [A:B] of order 3 and see if any of these have determinants not equal to zero. For example, a square sub matrix of the augmented matrix [A:B] is

$$[A]=\begin{bmatrix}25 & 5 & 1\\ 64 & 8 & 1\\ 89 & 13 & 2\end{bmatrix}$$

has $\det(A)=0$. This means, we need to explore other square sub matrices of order 3 of the augmented matrix [A:B] and find their determinants.

That is, $[E]=\begin{bmatrix}5 & 1 & 106.8\\ 8 & 1 & 177.2\\ 13 & 2 & 284.0\end{bmatrix}$

$\det(E)=0$

$$[F]=\begin{bmatrix}25 & 5 & 106.8\\ 64 & 8 & 177.2\\ 89 & 13 & 284.0\end{bmatrix}$$

$\det(F)=0$

$$[G]=\begin{bmatrix}25 & 1 & 106.8\\ 64 & 1 & 177.2\\ 89 & 2 & 284.0\end{bmatrix}$$

$\det(G)=0$

All the square sub matrices of order 3×3 of the augmented matrix [A:B] have a zero determinant. So the rank of the augmented matrix [A:B] is less than 3. Is the rank of augmented matrix [A:B] equal to 2?. One of the 2×2 sub matrices of the augmented matrix [A:B] is

$$[H]=\begin{bmatrix}25 & 5\\ 64 & 8\end{bmatrix}$$

and

$\det(H)=-120\neq0$

So the rank of the augmented matrix [A:B] is 2.

Now we need to find the rank of the coefficient matrix [A:B] .

$$[A]=\begin{bmatrix}25 & 5 & 1\\ 64 & 8 & 1\\ 89 & 13 & 2\end{bmatrix}$$

And $\det(A)=0$

So the rank of the coefficient matrix $[A]$ is less than 3. A square sub matrix of the coefficient matrix $[A]$ is $[J]=\begin{bmatrix}5 & 1\\ 8 & 1\end{bmatrix}$, $\det(J)=-3\neq0$

So the rank of the coefficient matrix $[A]$ is 2.

Hence, rank of the coefficient matrix $[A]$ equals the rank of the augmented matrix [A:B] . So the system of equations $[A][X]=[C]$ is consistent.

Example 7 : Use the concept of rank to find the solution , if $\begin{bmatrix}25 & 5 & 1\\ 64 & 8 & 1\\ 89 & 13 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}106.8\\ 177.2\\ 280.0\end{bmatrix}$ is consistent or inconsistent.

Solution : The augmented matrix is $[A:B]=\begin{bmatrix}25 & 5 & 1 & :106.8\\ 64 & 8 & 1 & :177.2\\ 89 & 13 & 2 & :280.0\end{bmatrix}$

Since there are no square sub matrices of order 4×4 as the augmented matrix [A:B] is a 4×3 matrix, the rank of the augmented matrix[A:B] is at most 3. So let us look at square sub matrices of the augmented matrix [A:B] of order 3 and see if any of the 3×3 sub matrices have a determinant not equal to zero. For example, a square sub matrix of order 3×3 of [A:B]

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$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 89 & 13 & 2 \end{bmatrix}$$

$\det(D) = 0$, So it means, we need to explore other square sub matrices of the augmented matrix $[B]$

$$[E] = \begin{bmatrix} 5 & 1 & 106.8 \\ 8 & 1 & 177.2 \\ 13 & 2 & 280.0 \end{bmatrix}$$

$\det(E) = 12.0 \neq 0$.

So the rank of the augmented matrix $[B]$ is 3.

The rank of the coefficient matrix $[A]$ is 2 from the previous example.

Since the rank of the coefficient matrix $[A]$ is less than the rank of the augmented matrix [A:B] , the system of equations is inconsistent. Hence, no solution exists for $[A][X] = [C]$.

If a solution exists, how do we know whether it is unique?

In a system of equations $[A][X] = [C]$ that is consistent, the rank of the coefficient matrix $[A]$ is the same as the augmented matrix $[A|C]$. If in addition, the rank of the coefficient matrix $[A]$ is same as the number of unknowns, then the solution is unique; if the rank of the coefficient matrix $[A]$ is less than the number of unknowns, then infinite solutions exist.

Example : We found that the following system of equations

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

is a consistent system of equations. Does the system of equations have a unique solution or does it have infinite solutions?

Solution: The coefficient matrix is $[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$ and the right hand side is $[C] = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$

While finding out whether the above equations were consistent in an earlier example, we found that the rank of the coefficient matrix (A) equals rank of augmented matrix $[A \vdots C]$ equals 3.

The solution is unique as the number of unknowns = 3 = rank of (A).

Example : Prove that the following system of equations $\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 89 & 13 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 284.0 \end{bmatrix}$

is a consistent system of equations. Is the solution unique or does it have infinite solutions?

Solution: While finding out whether the above equations were consistent, we found that the rank of the coefficient matrix $[A]$ equals the rank of augmented matrix $(A \vdots C)$ equals 2

Since the rank of $[A] = 2$ < number of unknowns = 3, infinite solutions exist.

Example: If we have more equations than unknowns in [A] [X] = [C], does it mean the system is inconsistent?

Solution: No, it depends on the rank of the augmented matrix $[A \vdots C]$ and the rank of $[A]$.

a) For example $\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \\ 89 & 13 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \\ 284.0 \end{bmatrix}$

is consistent, since

rank of augmented matrix = 3 = rank of coefficient matrix = 3

Now since the rank of (A) = 3 = number of unknowns, the solution is not only consistent but also unique.

b) For example $\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \\ 89 & 13 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \\ 280.0 \end{bmatrix}$ is inconsistent, since

rank of augmented matrix = 4 > rank of coefficient matrix = 3

c) For example $\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 50 & 10 & 2 \\ 89 & 13 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 213.6 \\ 280.0 \end{bmatrix}$ is consistent, since

rank of augmented matrix = 2 = rank of coefficient matrix = 2

But since the rank of $[A]$ = 2 < the number of unknowns = 3, infinite solutions exist.

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EXERCISE

Q4. Solve the system of equations $3x+3y+2z=1, x+2y=4, 10y+3z=-2, 2x-3y-z=5$ **[May 2019]**

Q5. Solve the system of equations $2x+3y+4z=11, x+5y+7z=15, 3x+11y+13z=25$ **[May 2018]**

Q6. Show that the following system of equation is inconsistent:
$x-2y+z-w = -1$, $3x-2z+3w=-4$, $5x-4y+w=-3$ **[June15]**

Q7. Test the consistency of the following system of equations and solve it if possible:
$4x-5y+z=2$, $3x+y-2z=9$, $x+4y+z=5$ **Ans**. x=2,y =1 ,z =-1

Q8. Investigation for what value of μ & λ the following equations
$x + y + z=6$, $x+2y+3z=10$, $x+2y+\lambda z= \mu$

Q1. no solution (2) a unique solution (3) an infinite solution. **Ans. μ=10 , λ=3,**
[June, Dec.05,June 08,March10, June15, June 16 (CBCS)]

Q9. Investigation for what value of μ & λ the following equations
$2x+3y+5z=9$, $7x+3y-2z =8$, $2x+3y+\lambda z= \mu$
(1) no solution (2) a unique solution (3) an infinite solution. **Ans:** λ=5 ,9 ,

Q10. Find out for what value of k the equation
$5x+3y+7z=4$, $3x+26y+2z=9$, $7x+2y+10z=5$
$2x+y+2z=0, 2x+y+3z=0, 4x+3y+bz=0$
have a solution and solve it in each case. **Ans. (i) k=1, Infinite Sol. (ii)k=2 , Infinite Sol.**
[Dec. 05, Dec.2014]

Q11. Examine the system of equations $5x+3y+14z=4$, $y+2z =2$, $x-y+2z=0$, $2x+y +6z=2$
is consistent also find solution if it is consistent . **Ans. x=-2,y =0 ,z =1**

Q12. Test the consistency of the following equation and solve it
$5x+3y+7z=4$, $3x+26y+2z=9$, $7x+2y+10z=5$ **Ans. *x=5-16k ,y =k ,z =11k-3,***
[June 03, Dec. June 08, April 09, June 2011, Dec. 2011, Dec. 2012, Nov. 18]

Q13. Examine the system of equations for consistency and solve if consistent:
$x+2y-z=33x- y+2z =1$, $2x-2y+3z=2$, $x-y +z=-1$ **Ans. *x =-1, y =4 ,z =4*** **[Dec.07]**

Q14. Test the consistency of the following equation and solve it,*x+2y-5z=-9, 3x-y+2z=5, 2x+3y-z=3* .
Ans: x=1/2,y=3/2,z=5/2 **[Dec. 2013]**

Q15. Solve the system of equations : $x+3y-2z=0$, $2x-y+4z=0$, $x-11y+14z=0$
Ans: x=-10k/7, y=8k/7, z=k **[June 2014]**

Q16. Determine the value of ***b*** , such that system of homogeneous equations has (i)Trivial sol. (ii) non- trivial Sol. Also find non trivial solution.: $2x+y+2z=0, 2x+y+3z=0, 4x+3y+bz=0$
Ans: x=k, y=-4k, z=k **[June 2012]**

1.20 EIGENVALUE AND EIGENVECTORS:

The word eigenvalue comes from the German word Eigenwert where Eigen means characteristic and Wert means value.

This is an important part of linear algebra because it has many applications in the areas of physical sciences and engineering. This section is straightforward but it does rely on a number of topics in linear algebra such as matrices, determinants, vectors etc.

You need to thoroughly know how to evaluate determinants to understand this chapter.

1.21 Definition of Eigenvalue and Eigenvector :

If $[A]$ is a $n \times n$ matrix, then $[X] \neq \vec{0}$ is an eigenvector of $[A]$ if $[A][X] = \lambda[X]$

where λ is a scalar and $[X] \neq 0$. The scalar λ is called the eigenvalue of $[A]$ and $[X]$ is called the eigenvector corresponding to the eigenvalue λ.

Eigenvectors are not unique in the sense that any eigenvector can be multiplied by a constant to form another eigenvector. For each eigenvector there is only one associated eigenvalue, however.

In order to find the eigenvectors of a matrix we must start by finding the *eigenvalues*. To do this we take everything over to the LHS of the Characteristic equation (Eigen equation) :

$$\mathbf{Ax} - \lambda\mathbf{x} = \mathbf{0},$$

then we pull the vector $\mathbf{x}$ outside of a set of brackets:

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}.$$

The only way this can be solved is if $\mathbf{A} - \lambda\mathbf{I}$ does not have an inverse[1], therefore we find values of λ such that the determinant of $\mathbf{A} - \lambda\mathbf{I}$ is zero:

$$|\mathbf{A} - \lambda\mathbf{I}| = 0.$$

Once we have a set of eigenvalues we can substitute them back into the original equation to find the eigenvectors. As always, the procedure becomes clearer when we try some examples:

Example : Determine the eigenvalues of $\mathbf{A} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 4 & 0 \\ 3 & 5 & -3 \end{pmatrix}$

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Solution:

$$\det(\mathbf{A}-\lambda\mathbf{I})=\det\begin{pmatrix}1-\lambda & 0 & 4\\ 0 & 4-\lambda & 0\\ 3 & 5 & -3-\lambda\end{pmatrix} \quad \leftarrow \text{Row 2}$$

$$=(4-\lambda)\left[\det\begin{pmatrix}1-\lambda & 4\\ 3 & -3-\lambda\end{pmatrix}\right] \qquad \left[\begin{array}{l}\text{Expanding the}\\ \text{Second Row}\end{array}\right]$$

$$=(4-\lambda)\left[(1-\lambda)(-3-\lambda)-(3\times 4)\right] \qquad [\text{By Determinant of 2 by 2}]$$

$$=(4-\lambda)\left[(\lambda-1)(3+\lambda)-12\right] \qquad [\text{Taking Out Minus Signs}]$$

$$=(4-\lambda)\left[3\lambda+\lambda^2-3-\lambda-12\right] \qquad [\text{Opening Brackets and } --=+]$$

$$=(4-\lambda)\left[\lambda^2+2\lambda-15\right] \qquad [\text{Simplifying}]$$

$$=(4-\lambda)(\lambda+5)(\lambda-3) \qquad [\text{Factorizing}]$$

By the characteristic equation , which is $\det(\mathbf{A}-\lambda\mathbf{I})=0$, means that we equate all the above to zero:

$$(4-\lambda)(\lambda+5)(\lambda-3)=0$$

Solving this gives the eigenvalues $\lambda_1=4,\ \lambda_2=-5$ and $\lambda_3=3$.

Example : Determine the eigenvectors associated with $\lambda_3=3$ for the matrix $\mathbf{A}=\begin{pmatrix}1 & 0 & 4\\ 0 & 4 & 0\\ 3 & 5 & -3\end{pmatrix}$

Solution. Substituting the eigenvalue $\lambda_3=\lambda=3$ and the matrix $\mathbf{A}=\begin{pmatrix}1 & 0 & 4\\ 0 & 4 & 0\\ 3 & 5 & -3\end{pmatrix}$ into $(\mathbf{A}-\lambda\mathbf{I})X=\mathbf{O}$

(subtract 3 from the leading diagonal) gives:

$$(\mathbf{A}-3\mathbf{I})X=\begin{pmatrix}1-3 & 0 & 4\\ 0 & 4-3 & 0\\ 3 & 5 & -3-3\end{pmatrix}X=\mathbf{O}$$

Where ***X*** is the eigenvector corresponding to $\lambda_3=3$.

Remark: *What is the zero vector, **O**, equal to?*

Remember this zero vector is $\mathbf{O}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ and also let $X=\begin{pmatrix}x\\y\\z\end{pmatrix}$. Substituting these into the above and

simplifying gives

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$$\begin{pmatrix} -2 & 0 & 4 \\ 0 & 1 & 0 \\ 3 & 5 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Expanding this yields the linear system

$$-2x + 0 + 4z = 0 \qquad (1)$$
$$0 + y + 0 = 0 \qquad (2)$$
$$3x + 5y - 6z = 0 \qquad (3)$$

From the middle equation (2) we have $y = 0$. From the top equation (1) we have

$$2x = 4z \quad \text{which gives } x = 2z$$

If $z = 1$ then $x = 2$; or more generally if $z = t$ then $x = 2t$ where $t \neq 0$ [Not Zero].

The general eigenvector $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2t \\ 0 \\ t \end{pmatrix} = t\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ where t and corresponds to $\lambda_3 = 3$.

Similarly we can find the eigenvectors corresponding to $\lambda_1 = 4$ and $\lambda_2 = -5$.

1.22 Some theorems of eigenvalue and eigenvectors.

Theorem 1: If $[A]$ is a $n \times n$ triangular matrix – upper triangular, lower triangular or diagonal, the eigen values of $[A]$ are the diagonal entries of $[A]$.

Theorem 2: $\lambda = 0$ is an eigenvalue of $[A]$ if $[A]$ is a singular (noninvertible) matrix.

Theorem 3: $[A]$ and $[A]^T$ have the same eigenvalue.

Theorem 4: Eigenvalue of a symmetric matrix are real.

Theorem 5: Eigenvectors of a symmetric matrix are orthogonal, but only for distinct eigenvalue.

Theorem 6: $|\det(A)|$ is the product of the absolute values of the eigenvalue of $[A]$.

EXERCISE

Q1. If A is non singular matrix . Show that the Eigen value of A^{-1} , are the reciprocal of Eigen values of A. **[June 14]**

Q2. Prove that the Eigen values of a symmetric matrix are real.

Q3. Prove that the Eigen values of a hermitian matrix are real.

Q4. Prove that the Eigen values of an idempotent matrix are either zero or unity. **[June 2007]**

Q5. Prove that the sum of the Eigen values of a square matrix is equal to the sum of its principal

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diagonal.

Q6. Prove that the product of all Eigen values of A is equal to its determinants. **[Dec. 2014]**

Q7. Find the Eigen values & Eigen vector of the matrix $A=\begin{bmatrix}3 & 2\\ -1 & 0\end{bmatrix}$ **Ans** .λ=1,2

Q8. Find the Eigen values & Eigen vector of the matrix $A=\begin{bmatrix}1 & -2\\ -5 & 4\end{bmatrix}$ **Ans** .λ=6,-1,

Q9. Find the sum and product of Eigen values of the matrix $A=\begin{bmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{bmatrix}$

Ans: Sum=12, Product=36 **[Dec.14,]**

Q10. Find the Eigen values of $A=\begin{bmatrix}3 & -4 & 4\\ 1 & -2 & 4\\ 1 & -1 & 3\end{bmatrix}$. Also find the Eigen values of A^{-1} and A^2.

Ans :λ= -1,2,3 (ii) for A^{-1}: λ= -1, 1/2,1/3 (iii) for A^2: λ= 1,4,9 **[June 2012]**

Q11. Find the Eigen values & Eigen vector of the matrix $A=\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}$

Ans .λ=1,-1 **[June 2006]**

Q12. If 3 and 15 are two Eigen values of $A=\begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}$. Find the remaining Eigen value and $|A|$.

Ans: λ=0 , $|A|$=0 **[June 08]**

Q13. Find the Eigen values & Eigen vector of the matrix $A=\begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}$

Ans .λ=0,3,15 [Dec.04, Jan. 08, Dec.12]

Q14. Find the Eigen values & Eigen vector of the matrix $A=\begin{bmatrix}3 & 1 & 4\\ 0 & 2 & 6\\ 0 & 0 & 5\end{bmatrix}$ **Ans .λ=2,3,5** **[Dec. 2010]**

Q15. Find the Eigen values & Eigen vector of the matrix $A=\begin{bmatrix}1 & 2 & 2\\ 0 & 2 & 1\\ -1 & 2 & 2\end{bmatrix}$ **[June 17 CBCS]**

Q16. Find the Eigen values & Eigen vector of the matrix $A=\begin{bmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{bmatrix}$ **Ans** **[May 2019]**

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Q17. Find the Eigen values & Eigen vector of $A=\begin{bmatrix}2&2&1\\1&3&1\\1&2&2\end{bmatrix}$ **Ans :λ= 1 ,1, 5 [Dec. 05,Dec. 13,14,June 15]**

Q18. Find the Eigen values & Eigen vector of the matrix $A=\begin{bmatrix}-2&2&-3\\2&1&-6\\-1&-2&0\end{bmatrix}$

Ans : λ= -3,-3,5[March10,June13, June14]

Q19. Find the Eigen values & Eigen vector of the matrix $A=\begin{bmatrix}2&1&1\\1&2&1\\0&0&1\end{bmatrix}$

Ans: λ= 1,1,3 [June 04, 07, Dec. 11,18]

Q20. Find the Eigen values & Eigen vector of the matrix $A=\begin{bmatrix}6&-2&2\\-2&3&-1\\2&-1&3\end{bmatrix}$

Ans :λ= 2 ,2 ,8 [Dec.03,June08,April 09, Jan. 17]

Q21. Find the Eigen values & Eigen vector of the matrix $A=\begin{bmatrix}1&0&0\\0&3&-1\\0&-1&3\end{bmatrix}$ **[June16(CBCS)]**

1.22 CAYLEY HAMILTON THEOREM

Every square matrix satisfies its own characteristic equation.

Let $A = [a_{ij}]_{n\times n}$ be a square matrix

$$\text{then, } A=\begin{bmatrix}a_{11}&a_{12}&\dots&a_{1n}\\a_{21}&a_{22}&\dots&a_{2n}\\\dots&\dots&\dots&\dots\\a_{n1}&a_{n2}&\dots&a_{nn}\end{bmatrix}_{n\times n}$$

Then characteristic polynomial of A

$$\varphi(\lambda)=A-\lambda I=\begin{bmatrix}a_{11}-\lambda&a_{12}&\dots&a_{1n}\\a_{21}&a_{22}-\lambda&\dots&a_{2n}\\\dots&\dots&\dots&\dots\\a_{n1}&a_{n2}&\dots&a_{nn}-\lambda\end{bmatrix}$$

The characteristic equation is $|A-\lambda I|=0$

$$\Rightarrow p_0\lambda^n+p_1\lambda^{n-1}+p_2\lambda^{n-2}+\dots+p_n=0$$

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We have to prove that $p_0A^n + p_1A^{n-1} + p_2A^{n-2} + ... + p_n = 0$ ……….(1)

To find A^{-1} :- Pre multiplying equation (1) by A^{-1} , we have

$$0 = p_0A^{n-1} + p_1A^{n-2} + p_2A^{n-3} + ... + p_{n-1}I + p_nA^{-1}$$
$$\Rightarrow A^{-1} = -\frac{1}{p_n}[p_0A^{n-1} + p_1A^{n-2} + p_2A^{n-3} + ... + p_{n-1}I]$$

This result gives the inverse of A in terms of (n-1) powers of A and is considered as a practical method for the computation of the inverse of the large matrices

If m is a positive integer such that m > n then any positive integral power A^m of A is linearly expressible in terms of those of lower degree.

EXAMPLES

Example: Verify Cayley – Hamilton theorem for the matrix A = $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$. Hence compute A^{-1} .

[Nov. 2018]

Solution : The characteristic equation of A is

$$|A - \lambda I| = 0 \quad \text{i.e.,} \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

or $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$ (*on simplification*)

To verify Cayley – Hamilton theorem, we have to show that $A^3 - 6A^2 + 9A - 4I = 0$ … (1)

Now $$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -22 & -21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$A^3 - 6A^2 + 9A - 4I = 0$

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$$\begin{bmatrix}22 & -22 & -21\\ -21 & 22 & -21\\ 21 & -21 & 22\end{bmatrix} - 6\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix} + 9\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix} - 4\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} = 0$$

This verifies Cayley – Hamilton theorem.

To find A^{-1} : Now, pre – multiplying both sides of (1) by A^{-1}, we have $A^2 - 6A + 9I - 4A^{-1} = 0$

=> $4A^{-1} = A^2 - 6A + 9I$

$$\Rightarrow 4A^{-1} = \begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix} - 6\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix} + 9\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}3 & 1 & -1\\ 1 & 3 & 1\\ -1 & 1 & 3\end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{4}\begin{bmatrix}3 & 1 & -1\\ 1 & 3 & 1\\ -1 & 1 & 3\end{bmatrix}$$

EXERCISE

Q22. State and prove Cayley-Hamilton Theorem. **[Dec. 2010]**

Q23. Verify Cayley –Hamilton theorem for the matrix $A = \begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$ verify Cayley –Hamilton theorem and find A^{-1}.

Ans. $A^3 - 6A^2 + 9A - 4I = 0$, $A^{-1} = \frac{1}{4}\begin{bmatrix}3 & 1 & -1\\ 1 & 3 & 1\\ -1 & 1 & 3\end{bmatrix}$ **[Dec.02, 08,June11,12]**

Q24. Verify Cayley –Hamilton theorem for the matrix $A = \begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 5\\ 3 & 5 & 6\end{bmatrix}$ verify Cayley –Hamilton theorem and find A^{-1}

Ans. $A^3 - 11A^2 - 4A + I = 0$, $A^{-1} = \begin{bmatrix}1 & -3 & 2\\ -3 & 3 & -1\\ 2 & -1 & 0\end{bmatrix}$ **[June 2015, 17 CBCS]**

Q25. Verify Cayley –Hamilton theorem $A = \begin{bmatrix}1 & 2 & -2\\ 1 & 1 & 1\\ 1 & 3 & -1\end{bmatrix}$ **[May 2019]**

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Q26. Verify Cayley –Hamilton theorem and find A^{-1}. Given $A=\begin{bmatrix}1 & 0 & 2\\0 & 2 & 1\\2 & 0 & 3\end{bmatrix}$ **[June16 (CBCS)]**

Q27. Verify Cayley –Hamilton theorem for the matrix $A=\begin{bmatrix}1 & 3 & 7\\4 & 2 & 3\\1 & 2 & 1\end{bmatrix}$ verify Cayley –Hamilton theorem and find A^{-1}

Ans. $A^3-4A^2-20A-35I=0$, $A^{-1}=\frac{1}{35}\begin{bmatrix}-4 & -11 & -5\\-1 & -6 & 25\\6 & 1 & -10\end{bmatrix}$ **[June04, Dec. 04]**

Q28. Find the characteristic equation of matrix $A=\begin{bmatrix}0 & 0 & 1\\3 & 1 & 0\\-2 & 1 & 4\end{bmatrix}$ verify Cayley –Hamilton theorem and find A^{-1}

Ans: $A^3-5A^2+6A-11I=0$, $A^{-1}=\frac{1}{5}\begin{bmatrix}4 & 1 & -1\\-12 & 2 & 3\\5 & 0 & 0\end{bmatrix}$ **[June 2013]**

Q29. Find the characteristic equation of matrix $A=\begin{bmatrix}1 & 2 & -1\\0 & 1 & -1\\3 & -1 & 1\end{bmatrix}$ verify Cayley –Hamilton theorem and find A^{-1}

Ans. $A^3-3A^2+5A+3I=0$, $A^{-1}=-\frac{1}{3}\begin{bmatrix}0 & -1 & -1\\-3 & 4 & 1\\-3 & 7 & 1\end{bmatrix}$ **[June09, Dec. 2011]**

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